Rutgers University: Algebra Written Qualifying Exam

January 2005: Day 1 Problem 8 Solution

Exercise. Prove that no group of order 30 is simple.

Solution.

Let G be a group of order 30. We want to show that there is a normal subgroup of G that is ***not*** $\{e\}$ or G.

$$30 = 2 \cdot 3 \cdot 5$$

By the third Sylow theorem,

$$\begin{array}{lllll} n_3 \equiv 1 \mod 3 & \text{and} & n_3 \mid 10 & \implies & n_3 = 1 \text{ or } 10 \\ n_5 \equiv 1 \mod 5 & \text{and} & n_5 \mid 6 & \implies & n_2 = 1 \text{ or } 6 \end{array}$$

If there is only $n_3 = 1$ Sylow 3-subgroups, then the Sylow 3 subgroup is normal in G by the second Sylow theorem.

$\implies$ G is not simple

Otherwise, $n_3 = 10$, and G has $10(3-1) = 20$ elements of order 3.

There are $30 - 20 = 10$ remaining elements in G

$\implies$ there cannot be $6(5-1) = 24$ elements of order 5

$\implies$ $n_5 \neq 6$

$\implies$ $n_5 = 1$

$\implies$ there must be only one Sylow 5-subgroup

$\implies$ the Sylow 5 subgroup is normal in G by the second Sylow theorem

$\implies$ G is not simple

Thus, no group of order 30 is simple.